

A rigidity result on the ideal boundary structure of smooth space-times

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Abstract. Following a survey of the abstract boundary definition of Scott and Szekeres, a rigidity result is proved for the smooth case, showing that the topological structure of the regular part of this boundary is invariantly defined.

1. Introduction

The most intuitively natural definition of a boundary to a space-time (M, g) would seem to be obtained by topologically embedding M in a larger manifold \widehat{M} of the same dimension by means of a map $\phi : M \rightarrow \widehat{M}$, followed by taking the topological boundary $\partial_\phi M \stackrel{\text{def}}{=} \partial(\phi(M))$ of the image $\phi(M)$ in \widehat{M} . (Where ϕ is of differentiability class C^ℓ , $\ell \geq 1$ we shall refer to this as a C^ℓ *envelopment*, or simply a *smooth envelopment* if the class C^ℓ is understood.) The problem with this is that, unless further conditions are imposed, $\partial_\phi M$ depends critically on the choice of ϕ , the different possible $\partial_\phi M$ for different ϕ being so variable in structure that they contain no invariant information.

Because of this difficulty, this approach was abandoned early on the history of relativity, in favour of boundary constructions linked more closely to the detailed geometry of M . If, however, we were to restrict ourselves to *regular envelopments*, meaning envelopments where ϕ was a metric isometry into a “regular” (in some sense) (\widehat{M}, \hat{g}) , then we might hope for more rigidity (that is, constancy of topological structure) in $\partial_\phi M$. To provide an overall context for this and related constructions Scott and Szekeres [1] have developed the *abstract boundary construction* which incorporates all possible boundaries $\partial_\phi M$ with appropriate identifications being made between corresponding subsets of the boundaries (boundary sets). It is already known that if the metric is used certain topological properties of boundary sets *are* preserved under

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equivalence, even allowing for quite unpleasant properties of the metric [2]. This work cited can be said to serve as an interesting example of the relative ease with which certain simple results about the abstract boundary can be obtained, and also the crucial rôle played by our “regularity/extendibility” assumptions (q.v.); however, the work is not used here.

In this paper we review the abstract boundary construction, we survey possible definitions of the notion of “regular”, and we show within this framework that we can achieve an appropriate rigidity of structure for a boundary constructed from regular envelopments. More precisely, we show that if we consider only boundaries that are regular and satisfy a Lipschitz condition, then all representatives of an equivalence class of boundary sets, in the sense of the abstract boundary, are homeomorphic.

In what follows, by a *pseudo-Riemannian manifold* (M, g) we will mean a Hausdorff, paracompact topological manifold M , without a boundary in the usual sense, that is equipped with a C^1 atlas and a metric g .¶ If, further, M is connected, $n = \dim M \geq 2$, and the metric g is of Lorentz signature $(-+\cdots+)$, then we shall refer to (M, g) as a *space-time*. If g is C^ℓ , some $\ell \geq 1$, the covariant differential with respect to the unique torsion-free metric-compatible connection on M will be denoted ∇ , or $\overset{g}{\nabla}$ if we wish to emphasise the rôle of the metric. In fact, we mostly use the notation of [3]. For example, $X_p \in T_p(M)$ will denote the value of the vector field X at the point $p \in M$.

In referring to envelopments a *boundary set* is a subset $B \subset \partial_\phi$. (We will normally have primed objects belonging to an envelopment (M, \widehat{M}', ϕ') , with corresponding unprimed objects belonging to an envelopment (M, \widehat{M}, ϕ) .)

2. Regular abstract boundaries

The equivalence relation used in the definition of the abstract boundary is as follows. Suppose that we are given boundary sets B, B' of two envelopments (M, \widehat{M}, ϕ) , (M, \widehat{M}', ϕ') , respectively. We say that B *covers* B' ($B \triangleright B'$) if for every open neighbourhood \mathcal{W} of B in \widehat{M} , there is an open neighbourhood \mathcal{W}' of B' in \widehat{M}' such that

$$\phi \circ (\phi')^{-1}(\mathcal{W}' \cap \phi'(M)) \subset \mathcal{W}. \quad (1)$$

It is easily seen that $B \triangleright B'$ if and only if “one cannot approach B' from within M without also approaching B ”. In a sense, then, this means that B is “bigger” than B' , and this is the reason for the notation. As an example, let

$$\widehat{M} = \widehat{M}' = \mathbb{R}^n, \quad (2)$$

$$M = \mathbb{R}^n \setminus \{\mathbf{0}\} \approx \mathbb{R} \times S^{n-1}, \quad (3)$$

$$\phi = \text{inclusion}, \quad (4)$$

$$\phi'(r, \Theta) = (r + 1, \Theta) \in \mathbb{R}^n \setminus \{0\}. \quad (5)$$

¶ Note that we require all manifolds to have differentiable structure, but make no demands on g at the moment, beyond continuity; note also that saying that (M, g) is C^k means that g is C^k , and the atlas of M is at least C^{k+1} .

Then $p = \mathbf{0} \in \partial_\phi$ covers *any* subset of $\partial_{\phi'}$.

One says that boundary sets B and B' are *equivalent*, $B \sim B'$, if they cover each other: $B \triangleright B'$ and $B \triangleleft B'$. The equivalence class containing the boundary set B is denoted $[B]$, and is called an *abstract boundary set*. If $[B]$ contains a singleton boundary set $\{p\}$, then $[B]$ may also be denoted by $[p]$, and is called an *abstract boundary point*.

The collection of all abstract boundary points constitutes the *abstract boundary* $\mathcal{B}M$ of M ; that is,

$$\mathcal{B}(M) \stackrel{\text{def}}{=} \left\{ [p] : p \in \partial_\phi M \text{ for some envelopment } (M, \widehat{M}, \phi) \right\}.$$

According to this definition, boundary points and abstract boundary points admit quite a rich further classification [1], which need not concern us here. Note that the abstract boundary construction is not as developed (or, perhaps, useful!) as such constructions as the bundle boundary (*b*-boundary) of Schmidt [4, 5, 6, 7], the conformal boundary (*c*-boundary) of Geroch *et al.* [8, 9, 5], or even the “*A*-boundary” of Clarke [5]. The latter turns out to be very closely related to a certain (“Lipschitz”) *regular abstract boundary*.

We now introduce the rôle of a metric. Let (M, \widehat{M}, ϕ) be an envelopment of a C^k , $k \geq 1$ pseudo-Riemannian manifold (M, g) . A point $p \in \partial_\phi$, which has a neighbourhood \mathcal{U}_p such that $\widehat{M}_p \stackrel{\text{def}}{=} \phi(M) \cup \mathcal{U}_p$ can be endowed with a C^ℓ (some $1 \leq \ell \leq k$) pseudo-Riemannian metric \hat{g} extending g , i.e., $g = \psi^* \hat{g}$, is called *C^ℓ -regular*. The envelopment may then be referred to as $(M, g, \widehat{M}, \hat{g}, \phi)$.

For regular points we will use both the original setting of the abstract boundary [1] just described, and also a definition from earlier, unpublished work of C. J. S. Clarke and S. M. Scott on defining topologies on the abstract boundary. (Actually, the authors of [1] mention that their classification requires merely an affine connection, not necessarily the Levi-Civita connection of a pseudo-Riemannian metric.)

Definition 2.1 *Let \mathcal{E} be a collection of envelopments and*

$$X_\mathcal{E} \stackrel{\text{def}}{=} \sum_{\phi \in \mathcal{E}} \overline{\phi(M)} \quad (\text{a disjoint union}),$$

and define a relation \sim on $X_\mathcal{E}$ in the following way:

- (i) *If $x \in \partial_\phi$, $y \in \partial_\psi$, then \sim is the usual boundary set equivalence.*
- (ii) *If $x = \phi(\xi)$ with $\xi \in M$, then $y \sim x \iff y = \psi(\xi)$, some $\psi \in \mathcal{E}$.*

This is an equivalence relation, and we set $M_\mathcal{E} = X_\mathcal{E}/\sim$. Let π be the projection $X_\mathcal{E} \rightarrow M_\mathcal{E}$, identify

$$M \quad \text{with} \quad \{ [\phi(p)] : p \in M, \phi \in \mathcal{E} \},$$

and write $\partial_\mathcal{E} M$ (or $\partial_\mathcal{E}$) for $M_\mathcal{E} \setminus M$.

We will use the notation $\phi : M \rightarrow \widehat{M}_\phi$ for envelopments, and for later use will also assume, as is entirely reasonable, that each \widehat{M}_ϕ is assumed to have an atlas $\mathcal{A}_{\widehat{M}_\phi}$ for which the “pullback” $\phi^ \mathcal{A}_{\widehat{M}_\phi}$ is equivalent to the atlas \mathcal{A}_M on M . (We will impose a further condition on atlases in §4.)*

The key use of the above formalism is that it allows us to vary construction of the completion $M_{\mathcal{E}}$ as we vary our chosen set of envelopments \mathcal{E} . The usual abstract boundary is $\partial_{\mathcal{E}_0}M$, where \mathcal{E}_0 is the collection of *all* smooth envelopments ϕ of M into some smooth manifold \widehat{M}_ϕ . (In order to be absolutely correct mathematically, “all” needs to be qualified in such a way as to ensure that \mathcal{E} is a set and not a proper class—for instance by working in a category of concretely defined manifolds.)

In the next section we will survey in detail possible candidates for “regular” maps, metrics etc. in terms of a class Λ of maps between Euclidean spaces. For a given choice of Λ we will denote by $\mathbf{\Lambda}$ the corresponding category of class Λ maps between manifolds (e.g. $\mathbf{\Lambda} = \mathbf{C}^k$), and by $\Lambda(M, N)$ the set of morphisms of $\mathbf{\Lambda}$ from manifold M to manifold N .

By, e.g., the categorical statement “ $\mathbf{\Lambda} \subset \mathbf{C}^1$ ” we mean that $\Lambda(M, N) \subset C^1(M, N)$ for all M, N under consideration.

Armed with a suitable category of maps, then, we make an important set of definitions, the first of which generalises our earlier notion of “regularity” for $\mathbf{\Lambda} \supsetneq \mathbf{C}^1$, but is identical to the earlier notion for $\mathbf{\Lambda} = \mathbf{C}^1$.

Definition 2.2 *Let $\mathbf{\Lambda}$ be a category of maps as above, and*

$$g \in \Lambda(M, \text{PsR}_\nu^n(M)) \cap \Gamma(M, \text{PsR}_\nu^n(M))$$

($\Gamma(B, T)$ denotes the set of sections $B \rightarrow T$) be a given pseudo-Riemannian metric of signature $(n, n - \nu)$, i.e., a non-degenerate, symmetric bilinear form at every point of M . We shall say that g is Λ -extendible about p , where $p \in \partial_\phi$ for some envelopment $\phi \in \mathcal{E}_0$, if the following holds: there is a neighbourhood U_p of p (in \widehat{M}_ϕ) and a pseudo-Riemannian metric \hat{g}_p on U_p , $\hat{g}_p \in \Lambda(U_p, \text{PsR}_\nu^n(U_p))$, for which $\phi^\hat{g}_p = g$. Here $\text{PsR}_\nu^n(M)$ denotes the subbundle of the symmetric tensor product $\text{Symm}(T^*M \otimes T^*M)$ with fibres consisting of non-degenerate bilinear forms of the appropriate signature. Now let*

$$\mathcal{E}[\mathbf{\Lambda}] = \{ \phi \in \mathcal{E}_0 : \forall p \in \partial_\phi M, g \text{ is } \Lambda\text{-extendible about } p \},$$

$$\mathcal{R}[\mathbf{\Lambda}](M) = \{ [p]_\sim \in \partial_{\mathcal{E}_0}M : g \text{ is } \Lambda\text{-extendible about } p, \text{ where } p \in \partial_\phi \}.$$

We call this a regular abstract boundary.

Note that $\mathcal{R}[\mathbf{\Lambda}]$ can be identified, loosely at least, with the abstract boundary $\partial_{\mathcal{E}[\mathbf{\Lambda}]}M$ associated with the collection of all envelopments with “everywhere Λ -regular” boundaries.

One would expect these to be useful abstract boundaries in relativity theories, where we would like to seek properties of *all* possible metric extensions (of some class) through a “boundary point”, without using or requiring detailed knowledge of the analytic form of the metric involved.

3. Various regularity classes

We mention a few possible choices for our extension classes $\Lambda(\cdot)$, although we will only consider the first in this paper. For clarity we assume that we are speaking of Lorentzian metrics.

“Smooth regularity”, $\Lambda = C^k$, $k \geq 1$. Such regular abstract boundaries, the first to be considered, are the subject of unpublished work by C. J. S. Clarke and S. M. Scott.

“Existence theorem regularity.” There are several possibilities that suggest themselves immediately here, relying on the existence theorems of [10]. These rely on conditions such as the difference between the metric that we consider and a fixed “asymptotically flat” or “background” metric lying in the Sobolev space $H^{2.5+\varepsilon}(\mathbb{R}^{n-1})$, $\varepsilon > 0$ (the metrics checked are the restrictions of a full space-time metric to hypersurfaces). We could allow extensions such that through any point there is a spacelike hypersurface on which induced data satisfies the conditions; or such that through any point there is a set of space-like hypersurfaces whose normals at the point form an open set and which satisfy the conditions, or such that there is a (local or global) foliation satisfying the conditions.

“Non-quantum physics regularity.” It could be argued that a fairly robust criterion for pair production to become significant within a 3-dimensional region of “size” L where the components of the Riemann tensor in a “reasonable” frame are larger than R is that

$$R > \max(m^2 c^2 / \hbar^2, L^{-2})$$

the argument being either merely on dimensional grounds, or by requiring that virtual pairs acquire sufficient energy within the region during the Heisenberg uncertainty time to become non-virtual.

We would, then, require Λ to consist of metrics for which this condition did *not* hold, i.e., those metrics where we might expect Einstein’s theory to describe “reality” accurately.

“Distributional (Cauchy-Schwarz or Colombeau) regularity.” Here we would, technically, go outside our formalism and allow extensions—and presumably, although not necessarily, interior metrics also—that were not functions at all, but either Cauchy-Schwarz (linear) distributions, or Colombeau’s “generalised functions” (non-linear distributions). To cope with such generality, we would probably have to impose further conditions on the Levi-Civita connection components of these extended metrics, e.g., square-integrability in the first case and integrability in the second. Physically, these would describe, respectively, impulsive gravitational waves in their most obvious form, and “stringy” space-times admitting conical singularities.

“Geroch–Traschen regularity” [11], $\Lambda = C^0 \cap W_{loc}^{1,2}$ (intersection of categories of maps being defined by the intersection of function classes)

We call these *metrics of Geroch–Traschen type*. (and the second....) deleted
Physically, these appear to encompass gravitational wave space-times, though the metric must be transformed so that it is continuous. This “Rosen form” transformation typically destroys at least the C^1 structure of the manifold in question. We will propose a way to deal with this in a forthcoming publication

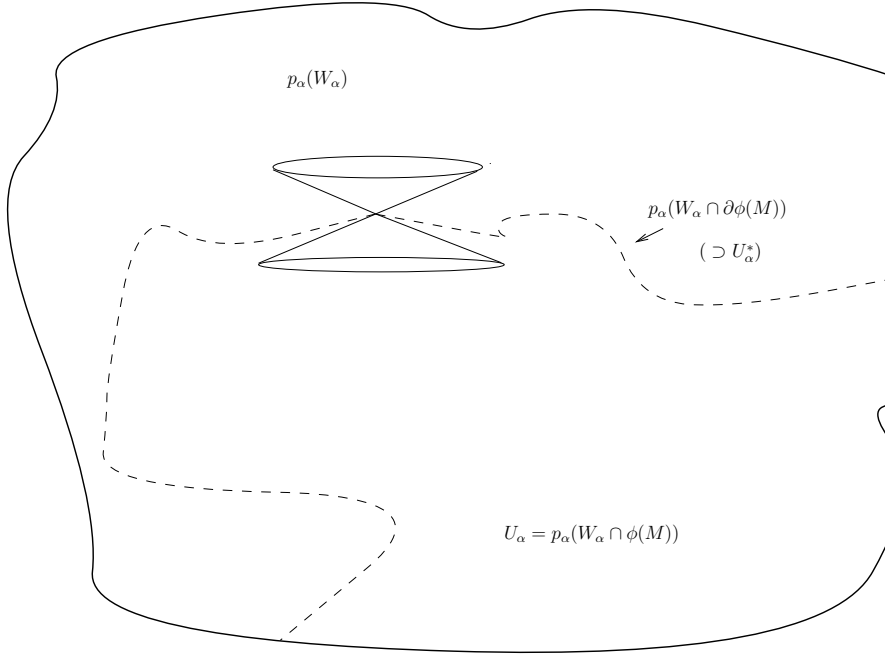


Figure 1. A typical Lipschitz boundary, on which a pair of proper (inward and outward) cones are shown.

(in which we will also allow degenerate metrics, though still requiring square-integrability of Levi-Civita connection components).

In this paper we consider only the first of these items.

4. Lipschitz boundaries

In this section we recall some material from [5], in which certain proscriptions on the nature of an atlas entail that the *A-boundary* can be constructed. This turns out to be a rather useful boundary construction for later work of Clarke and colleagues on singularity theorems, appearing in the same reference. It amounts to the requirement that the boundary “not wiggle about too much”...

Let $\mathcal{A} = \{(W_\alpha, p_\alpha)\}_{\alpha \in A}$ be an atlas for an envelopment \widehat{M}_ϕ of M , and write $U_\alpha = p_\alpha(W_\alpha \cap \phi(M)) \subset p_\alpha(W_\alpha) \subset \mathbb{R}^n$. For brevity, we will write U_α^* for that part of the boundary of the open set U_α , which, speaking somewhat loosely, corresponds to parts of “the boundary of M ” which “have two sides” and are “regular” with respect to our chosen g and Λ :

$$U_\alpha^*[g, \Lambda] \stackrel{\text{def}}{=} \left\{ x \in p_\alpha(\partial(\phi(M)) \cap W_\alpha) : g \text{ is } \Lambda\text{-extendible about } (p_\alpha)^{-1}(x) \right\} \setminus \left\{ x \in \partial U_\alpha : \text{a neighbourhood of } x \text{ is contained in } \overline{U_\alpha} \right\}, \quad (6)$$

See Figure 1.

Note that we exclude points of $\text{int } \bar{U}_\alpha$ from the definition of U_α^* . This is because portions of a boundary which are not “Lipschitz hypersurfaces” cannot make up the

“Lipschitz boundaries” that we define, following [5], in a moment. (In particular, this excludes higher codimension potential boundaries like the z -axis for $M = U = \mathbb{R}^3 \setminus \{z\text{-axis}\}$, $\phi = \text{inclusion}$.)

Finally, we impose a regularity condition on our chosen points U_α .

Definition 4.1 *We say that the envelopment \widehat{M}_ϕ has a Lipschitz boundary if the following three conditions hold, for each of the charts W_α (the last condition, depending on pairs of charts, is for the benefit of the A-boundary construction and will not concern us for the remainder of this work):*

(A_α) \bar{U}_α is compact in \mathbb{R}^n .

(B_α) There is a vector field $k_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (assumed to be as smooth as the atlases $\mathcal{A}, \mathcal{A}_\phi$), such that for each $x \in U_\alpha^*$, we have $|k_\alpha(x)| = 1$ and the existence of some $\delta = \delta(x) > 0$ with

$$y \in U_\alpha \iff k_\alpha(x) \cdot (y - x) < f_x(P_{k_\alpha(x)}(y - x))$$

if $\|y - x\| < \delta$. Here $\|\cdot\|$ denotes the Euclidean norm, f_x is some Lipschitz function $\mathbb{R}^n \rightarrow \mathbb{R}$, and $P_v y; y \mapsto y - (v \cdot y)v$ is the orthogonal projection from \mathbb{R}^n onto v^\perp .

($C_{\alpha\beta}$) Writing $\psi_{\alpha\beta} \stackrel{\text{def}}{=} p_\alpha \circ p_\beta^{-1}$ for the transition functions of \mathcal{A}_ϕ , $|D\psi_{\alpha\beta}|$ is bounded on $U_\alpha \cap U_\beta$.

Note that the choice $\phi = \text{identity}$, $\mathcal{A} = \mathcal{A}_\phi$ leads to a case of the A-boundary construction, where the subsets U_i^* of ∂U_i are taken to be the maximal such subsets through which g is Λ -extendible. (The union of M with the A-boundary $\bar{M} \setminus M$ constructed in [5] can be exhibited as a (C^0) manifold-with-boundary $\bar{M} \stackrel{\text{def}}{=} \bar{M}^{(A)}$ (the details are in the quoted reference).) We shall call this the *maximally regular (Clarke) A-boundary* construction.

We examine condition (B_α), the core of the matter, a little more closely. If $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a non-singular homothety (i.e., the composition of a dilation, translation, and rotation) with A_* taking the unit vector $k_\alpha(x)$ to $(0, \dots, 0, 1)$ and $A(x) = 0 \in \mathbb{R}^n$, then this is just the condition that

$$U_\alpha \cap B(0, \delta) = A^{-1} \left\{ z \in \mathbb{R}^n : \|z\| < \delta, z^n < \hat{f}_x(z^1, \dots, z^{n-1}) \right\}$$

for some Lipschitz $\hat{f}_x : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ with $\hat{f}_x(0) = 0$. It is clear, then, that this is equivalent to

$$U_\alpha^* \cap B(0, \delta) = A^{-1} \left\{ z \in \mathbb{R}^n : \|z\| < \delta, z^n = \hat{f}_x(z^1, \dots, z^{n-1}) \right\}.$$

Let \hat{K} be a Lipschitz constant for \hat{f}_x ; it follows that

$$\forall z \in U_\alpha^* \cap B(x, \delta(x)), \quad (z - \overset{\circ}{C}) \cap B(x, \delta) \subset U_\alpha \text{ and } (z + \overset{\circ}{C}) \cap B(x, \delta) \subset \mathbb{R}^n \setminus \bar{U}_\alpha,$$

where $\overset{\circ}{C} \stackrel{\text{def}}{=} \mathbb{R}^{>0} \cdot [B_{\mathbb{R}^{n-1}}(0, 1) \times \{\hat{K}\}]$ is the interior of a cone.

We give these cones $z \pm \overset{\circ}{C}$ names: $z - \overset{\circ}{C}$ (or rather, $(\psi_\alpha)^{-1} \left(A^{-1}(z - \overset{\circ}{C}) \right)$) we call a *proper inward cone* and that with “+” rather than “−” a *proper outward cone*.

Proposition 4.1 (Proposition 3.1, [5]) *Condition (B_α) of the definition of a Lipschitz boundary (Definition 4.1) is equivalent to the existence of proper inward and outward cones everywhere near U_α^* .*

Remark. In the reference quoted there is no reference to (what we call) proper outward cones—but *if condition (B_α) holds*, there must be a proper outward cone as well as a proper inward cone.

The existence of proper inward cones will be used in the next section. First we make a new definition.

Definition 4.2 *Consider the class of envelopments*

$$\mathcal{LE}[\Lambda] \stackrel{\text{def}}{=} \{ \phi \in \mathcal{E}[\Lambda] : \phi \text{ has a Lipschitz boundary} \},$$

and $\mathcal{LR}[\Lambda] \stackrel{\text{def}}{=} \partial_{\mathcal{LE}[\Lambda]}$ (denoted $\mathcal{LR}[M, g, \Lambda]$ if we want to highlight the manifold M and metric g), the abstract boundary constructed from this set of envelopments. We shall call the latter the Lipschitz regular abstract boundary. (Similarly we shall call the appropriate equivalence classes Lipschitz regular abstract boundary points, and their representatives Lipschitz regular boundary sets, and so on.)

Finally, we shall write $\mathcal{LR}^[M, g, \Lambda]$ for the class of all equivalents of boundary sets, without any demand that it contain a point as a representative. (This makes some discussions slightly easier.)*

5. The bundle metric—classical (C^1) case

If $\Lambda \subset \mathbf{C}^1$ and $g \in \mathbf{C}^1$ then there is a (continuous) connection on (M, g) and the frame bundle LM (structure group $GL(n, \mathbb{R})$) admits a topological metric d , induced by a C^0 Riemannian structure as in [5, Chap. 3], [7] or [6].

The same construction gives a topological metric \hat{d}_p on $L(\phi(M) \cup U_p)$ ($p \in \partial_\phi$), resulting from the (continuous) Levi-Civita connection of \hat{g}_p . However, there are at least two potential problems with this.

Firstly, the metric in the frame bundle depends on the boundary point: \hat{d}_p depends on the point p . This, however, will not prove to be a problem when we come to the crux of the matter, Theorem 5.2, but we must keep the subscript p in mind!

Secondly, we have the following matter which rather complicates our proceedings. For points $x, y \in \phi(M)$, we certainly have $\hat{d}_p(\tilde{x}, \tilde{y}) \leq d(\tilde{x}, \tilde{y})$ (tildes denoting points of fibres as usual)—loosely, $\hat{d}_p \leq d$ —but we may have $\hat{d}_p(\tilde{x}, \tilde{y}) < d(\tilde{x}, \tilde{y})$. This is evident from Figure 2 (take \hat{g}_p to be flat). Unfortunately, in a crucial argument below (Theorem 5.2) we will want to have $\hat{d}_p(\tilde{x}_i, \tilde{y}_i) \rightarrow 0 \Rightarrow d(\tilde{x}_i, \tilde{y}_i) \rightarrow 0$. The inequality “goes the wrong way”!

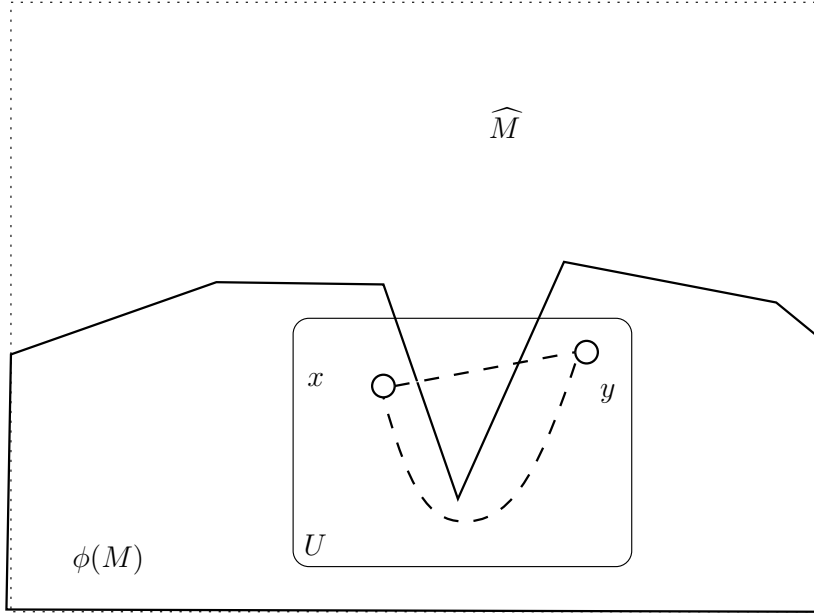


Figure 2. $\hat{d}(\tilde{x}, \tilde{y}) < d(\tilde{x}, \tilde{y})$ because there is a curve from x to y in \widehat{M}_ϕ of strictly shorter g.a.p. length than any in $\phi(M)$.

5.1. Using Lipschitz boundaries to imbed the Cauchy completion \overline{LM} in \widehat{LM}

It is quite sufficient for our needs, however, to have a mere bound “ $\phi_*d \leq \text{constant} \times \hat{d}_p$ ”, in a small neighbourhood of each point of the boundary. We will call into play the Lipschitz condition imposed on the boundary in the last section. But first, a technical (but straightforward) lemma. A sketched proof only will be given—the reader unsatisfied with this is invited to turn to the more rigorous, and general, treatment of a forthcoming paper. For now, though, note that we require that $\mathbf{A} \subset \mathbf{C}^1$.

Lemma 5.1 *Let g be a C^1 pseudo-Riemannian metric on \mathbb{R}^n . Then, given a chart $(\tilde{U}, \tilde{\varphi})$, a point $y \in \tilde{U}$, and any positive $\varepsilon > 0$, we can find a small neighbourhood $U \subset \tilde{U}$ of y in which $|\ell_1(\gamma) - \ell_2(\gamma)| < \varepsilon$ for all curves $\gamma : [0, 1] \rightarrow U$ whose images under $\tilde{\varphi}$ are straight lines. Here $\ell_1(\gamma) = k \times \int_\gamma \|\omega(\dot{\gamma})\|$, $\omega = \{w^a\}$ being a parallel frame along γ , is a g.a.p. length, k is a constant depending only on the frame at $\gamma(0)$, and $\ell_2(\gamma) = \int_\gamma \|\dot{\gamma}\|$ denotes Euclidean length.*

Proof sketch. Given y , we can choose U so small that $\sup_{x \in U, i, j} |g_{ij}((x) - \eta_{ij}|$ and $\sup_{x \in U, i, j, k} |g_{ij, k}((x))|$ are as small as desired (i.e., g is close to a pseudo-Euclidean metric in the C^1 strong Stiefel-Whitney topology [12]), so the Christoffel symbols of the metric g are as small as desired. We use the usual continuous dependance of the solutions of a (C^1) differential equation on its parameters (see, e.g., Theorem IV.2.1 of [13]), here specifying our curve by means of two extra parameters, namely its endpoints. The solution parallelly transported vector field depends continuously on the (small)

Christoffel symbols of g , so the solution is close to that for a flat metric, so for a short curve γ ,

$$\|\omega(\dot{\gamma})\| \approx c \times \|\dot{\gamma}\|$$

where c depends only on $\omega_{\gamma(0)}$. (Of course it depends on the metric g , but this is fixed once and for all.) This “closeness” is uniform for “endpoint parameters” in some neighbourhood $U \subset \mathbb{R}^n \times \mathbb{R}^n$ of the origin, by the continuous dependence cited. We get the result. \blacksquare

Theorem 5.2 *Consider an envelopment $\phi \in \mathcal{E}[\Lambda]$, $\Lambda \subset \mathbf{C}^1$, with a Lipschitz boundary, where $g \in \mathbf{C}^1$.*

Let $p \in \partial_\phi$. There is a neighbourhood U of p (in \widehat{M}_ϕ) such that we cannot have sequences $\{\tilde{x}_i\}, \{\tilde{y}_i\}$ of LM, lying over points of U , for which $\hat{d}_p(\phi_\tilde{x}_i, \phi_*\tilde{y}_i) \xrightarrow{i \rightarrow \infty} 0$ while $d(\tilde{x}_i, \tilde{y}_i)$ is bounded away from zero.*

Proof. Assume given sequences $\{\tilde{x}_i\}, \{\tilde{y}_i\}$ of LM for which $\hat{d}_p(\phi_*\tilde{x}_i, \phi_*\tilde{y}_i) \xrightarrow{i \rightarrow \infty} 0$. We require to show that $d(\tilde{x}_i, \tilde{y}_i) \xrightarrow{i \rightarrow \infty} 0$. By choosing a subsequence if necessary, we can assume that $\phi(x_i) \rightarrow \hat{x} \in \widehat{M}$, and hence that $\phi(y_i) \rightarrow \hat{x}$. From Proposition 4.1, if necessary by excluding a finite initial part of the sequence and applying a fixed homothety, we can ensure that there are inward cones of the form $w_i - \overset{\circ}{C}$, $z_i - \overset{\circ}{C}$, where in the coordinate chart (U_α, p_α) thus fixed we write $w_i = p_\alpha(\phi(x_i))$ and $z_i = p_\alpha(\phi(y_i))$. Recall that K is the Lipschitz constant in the definition of $\overset{\circ}{C}$. Let $\Pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ be the projection onto the first $n-1$ coordinates, and let e denote Euclidean distance in the coordinate system now chosen. Set

$$\begin{aligned} e_i &\stackrel{\text{def}}{=} e(w_i, z_i) \\ \Sigma_i &\stackrel{\text{def}}{=} \mathbb{R}^{n-1} \times \{w_i^n - (K+1)e_i\} \\ P_i &\stackrel{\text{def}}{=} \Pi((w_i - \overset{\circ}{C}) \cap \Sigma_i) \\ Q_i &\stackrel{\text{def}}{=} \Pi((z_i - \overset{\circ}{C}) \cap \Sigma_i) \end{aligned}$$

Then $p_\alpha(\hat{x}), z_i, w_i$ are on the same side of the hyperplane Σ_i , P_i is a ball of radius $(1+1/K)e_i$ and centre $w_i^* \stackrel{\text{def}}{=} \Pi(w_i)$, while Q_i is a ball of radius $((K+1)e_i + z_i^n - w_i^n)/K \geq e_i$ and centre $z_i^* \stackrel{\text{def}}{=} \Pi(z_i)$. Since $|w_i^* - z_i^*| \leq e_i$, these balls have a non-empty intersection containing some point $f_i^* = \Pi(f_i)$, say, where $f_i \stackrel{\text{def}}{=} (f_i^*, w_i^n - (K+1)e_i)$. Hence from Lemma 5.1

$$d(\tilde{x}_i, \tilde{y}_i) \leq \text{const} \times (e(w_i, f_i) + e(f_i, z_i)) \leq (2K+5)e_i.$$

But from Lemma 5.1 again, the far right hand side tends to zero, and so the result is proved. \blacksquare

Another, perhaps more satisfying, way of stating this result is as an “imbedding theorem”. (We use “imbedding” rather than “embedding” as a mnemonic for “injection”—there is no topological content to the statement.)

Corollary 5.3 *Let $\phi \in \mathcal{E}[\Lambda]$ have a Lipschitz boundary. For any boundary point $p \in \partial_\phi$, the map*

$$\overline{LM} \rightarrow \overline{\widehat{LM}_p} : \lim_{i \rightarrow \infty} (x_i, E_i) \mapsto \lim_{i \rightarrow \infty} (\phi(x_i), \phi_* E_i)$$

is one-one, i.e., it is an imbedding.

6. Application to regular abstract boundaries

We see that we can neither “coalesce” two regular boundary points into one without destroying the Lipschitz nature of the boundary near the points; similarly we cannot “tear” one Lipschitz regular boundary point into two:

Theorem 6.1 *Let $\Lambda \subset \mathbf{C}^1$ and $g \in \mathbf{C}^1$. A (regular) boundary set $B \subset \partial_\phi$, $\phi \in \mathcal{E}[\Lambda]$, of more than one point cannot be covered by a single (Lipschitz regular) point $q \in \partial_\psi$, $\psi \in \mathcal{LE}[\Lambda]$.*

Proof. Let p_i , $i = 1, 2$, be distinct points of B , assumed to be both covered by a single boundary point $q \in \partial_\psi$. We first deal with the first of the ‘potential problems’ mentioned just before §5.1. Of course we can construct a *single* (C^1) *coordinate chart* (U, τ) containing the two points, using the tubular neighbourhood theorem and a C^1 curve connecting the two points (the latter existing by connectedness of M). Since both these points are C^1 regular boundary points, then we may assume that $U_{p_1} \cap U_{p_2} = \emptyset$ (c.f. Definition 2.2), and thus that there is a *simultaneous* extension of g to $\widehat{M}_{p_1, p_2} \stackrel{\text{def}}{=} \phi(M) \cup U_{p_1} \cup U_{p_2}$. We may then construct the bundle metric \hat{d}_{p_1, p_2} as we did earlier when considering extensions about a *single* point. (The astute reader will have noted that a version of Theorem 5.2 using this new bundle metric will not be possible in general, for we cannot now shrink our neighbourhood to make “ $g_{ij} \approx \eta_{ij}$ ”. Fortunately, we only need to use Theorem 5.2 about the single (regular) point q . As a consequence of this *we do not need the boundary ∂_ϕ to be Lipschitz*, but only the boundary ∂_ψ about the single point q .)

Let $\{x_{i,j}\}_{j=1}^\infty$ be sequences of points of M for which $\phi(x_{i,j}) \xrightarrow{j \rightarrow \infty} p_i$, and $\tilde{x}_{i,j}$ be points of the fibres over $\phi(x_{i,j})$ in \widehat{LM}_ϕ . We shall write $\tilde{x}_{k,j} \stackrel{\text{def}}{=} (\phi(x_{k,j}), E_{k,j})$, $k = 1, 2$. Note that, as yet, we have said nothing about the fibre elements $\mathbf{E}_{k,j}$, $k = 1, 2$.

Now $\psi(x_{i,j})$, $i = 1, 2$, both converge to the covering point q . We can choose the fibre components of the $\tilde{x}_{i,j}$ so that these sequences also converge to the same point $\tilde{q} \in \widehat{LM}_\psi$. This means that \hat{d}_q -distance between the relevant points tends to zero, so by Theorem 5.2 we must have

$$d(\tilde{x}_{1,j}, \tilde{x}_{2,j}) \xrightarrow{j \rightarrow \infty} 0.$$

But $\hat{d}_{p_1, p_2}(\tilde{p}_1, \tilde{p}_2) = 2R > 0$, so $\hat{d}_{p_1, p_2}(\tilde{x}_{1,j}, \tilde{x}_{2,j}) > R$ for large j . Since “ $\hat{d}_{p_1, p_2} \leq d$ ”,

$$d(\tilde{x}_{1,j}, \tilde{x}_{2,j}) > R > 0$$

for all large enough j , in contradiction. ■

We arrive at a result sitting well with intuition.

Corollary 6.2 *Let $\Lambda \subset \mathbf{C}^1$ and $g \in \mathbf{C}^1$. Lipschitz regularity of boundary sets means that they “cannot be blown up further” without destroying regularity. To be more precise, consider two regular boundary sets $B_1 \subset \partial_\phi, B_2 \subset \partial_\psi$ of envelopments $\phi, \psi \in \mathcal{E}[\Lambda]$. Assume that the first envelopment ϕ has a Lipschitz boundary. Then no point of B_1 can cover **more than** one point of B_2 .*

If both envelopments have Lipschitz boundaries and $B_1 \sim B_2$, then each point of B_1 is equivalent to precisely one point of B_2 .

Proof. Let $p \in B_1$, and let $Q \stackrel{\text{def}}{=} \{ q \in B_2 : \text{if } q = \lim_j \psi(x_j), \text{ then } p = \lim_j \phi(x_j) \}$. Then Q is either empty, in which case p covers no subset of B_2 , or is the maximal subset of B_2 covered by p . In the latter case we apply Theorem 6.1, to get that Q must be a singleton. Thus no point of B_1 can cover more than one point of B_2 .

For the second part, note that by symmetry, the reverse of this statement is true, and the result follows. \blacksquare

The following lemma, although perhaps rather useless in other situations, does not *per se* depend on the Lipschitz, regular nature of the boundary sets to which we will apply it.

Lemma 6.3 *If $B_1 \sim B_2$ and the function $p \mapsto Q$ of Corollary 6.2 is singleton-valued (in which case we say $p \mapsto q$, if $Q = \{q\}$), and indeed maps each p to an equivalent q , then $B_1 \rightarrow B_2 : p \mapsto q$ is continuous.*

Proof. If not, then there is an open neighbourhood V of $q \in B_2$ such that there is no neighbourhood U of p all of whose boundary points correspond to points of V . If there are small enough neighbourhoods U intersecting B_1 only at p , then we get a trivial contradiction. So there is a sequence of boundary points $p_i \in B_1$, equivalent to points *outside* V , which converge to p .

But q covers p , so there is a sub-neighbourhood U' of U such that $\psi \circ \phi^{-1}(U' \cap \phi(M)) \subset V$. All sequences of points $\{x_j^i\}_j \subset \phi^{-1}(U' \cap \phi(M))$, with respective images under ϕ converging in j to the respective $p_i \in B_1$, must have images under ψ which converge to points outside of V . However, the “diagonal” sequence $\{x_i^i\}_i$ has $\phi(x_i^i) \xrightarrow{i \rightarrow \infty} p$, and so $\psi(x_i^i) \xrightarrow{i \rightarrow \infty} q$, in contradiction since $\widehat{M}_\psi \setminus V$ is closed. \blacksquare

By symmetry, we have the following.

Theorem 6.4 (Characterization of Lipschitz regular abstract boundary, smooth case) *If M is a differentiable manifold and $g \in \Lambda \subset \mathbf{C}^1$, then all representatives of an equivalence class of Lipschitz regular boundary sets $[B] \in \mathcal{LR}^*[M, g, \Lambda]$ are homeomorphic. That is, “the” topology of an equivalence class of boundary sets is independent of the particular envelopment used to define it, if all envelopments considered have Lipschitz, regular boundaries.*

Note that this means that every representative of a Lipschitz C^1 -regular abstract boundary point $[p] \in \mathcal{LR}[M, g, \Lambda]$ is itself a point. Of course, the nature of a Lipschitz boundary forces the original point p to be a point of an $(n - 1)$ -dimensional boundary hypersurface. As was alluded to earlier, it says that any envelopment of the space-time in which our original point is “blown up” to be represented by anything other than a single point destroys C^1 regularity of at least part of the new representative, *whether this new boundary is Lipschitz or not*.

This gives us a rather complete characterisation of those sections of boundaries of (e.g.) space-times through which the metric can be extended in a smooth (C^1) fashion. We are on firm *mathematical*, if not physical, ground when we suggest that each point of a Lipschitz boundary of a space-time through which the metric can be extended smoothly, has something to distinguish it from all others.

It is a natural question to ask whether the same (or a similar) characterisation is valid when we relax the smoothness assumption somewhat, allowing, e.g., gravitational shock waves. After all, we would certainly hope that the presence of gravitational waves in the space-time and extension thereof cannot “distort” the boundary so that the topological type of a portion of “boundary hypersurface”, through which extension is possible, could depend on the precise way in which the extension is performed.

Since boundaries are of physical interest precisely in situations of collapse, where strong gravitational fields are liable to generate shock- and impulse-waves, it is important to be able to relax as far as possible the smoothness assumptions in this work. It would be a problem indeed for our regular abstract boundaries if it turned out that “non-smooth perturbations” of a smooth metric could result in a space-time where, unlike the smooth case, the topological type of a portion of “regular boundary hypersurface” could depend on the precise way in which the extension is performed. In a forthcoming paper we will extend our results in this direction.

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